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Nonextensive thermostatistic properties of a q-generalized Fermi system

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Abstract. Based on the generalized statistic distribution derived from the Tsallis' entropy that has been successfully used in complex systems with long-range interactions and/or long-duration memory, the thermostatistic properties of a q-generalized Fermi system are studied. The total number of particles, internal energy, and heat capacity at constant volume are calculated for two different cases of $q \leq 1$ and $q \geq 1$, respectively, where q is an important parameter to evaluate the nonextensivity of the system. The thermostatistic characteristics of the system are discussed in detail. It is found from the results obtained here that the Fermi energy of such a system is independent of the parameter q and is equal to that of an original ideal Fermi system when $q \leq 1$, while other thermostatistic properties of the system depend closely on the parameter q. For example, when $q < 1$, the chemical potential of the system in certain region of temperature may be larger than the Fermi energy; when $q > 1$, some thermodynamic parameters of the system at low temperatures must be cut off.

PACS. 05.30.-d Quantum statistical mechanics – 05.20.-y Classical statistical mechanics – 05.70.-a Thermodynamics

1 Introduction

It is well known that the Fermi system is one of two types of the quantum systems in the natural world. It will show some unique quantum characteristics at very low temperatures. Specially, under suitable conditions the fermions may form Fermi pairs and become a Bose gas [1]. The Bose-Einstein condensation (BEC) may occur in such a system and was achieved recently by Markus Greiner et al. [2]. Thus, the investigation on Fermi gases has been an increasing interest subject over the past years.

Nonextensive statistical mechanics is based on the q-generalized entropy proposed by Tsallis [3] and developed by many researchers [4–14]. It has become a powerful tool to deal with some systems, which are more complex than a standard ideal gas and present long range interactions and/or long-duration memory. For example, it has been successfully used to study the properties of the generalized Bose system and a large number of significant results have been obtained [7–10]. Obviously, it is very meaning to investigate the properties of a generalized Fermi system by using nonextensive statistical mechanics.

In nonextensive statistical mechanics developed from the Tsallis' entropy, there are three different choices for the energy constraint. The first choice is very little used in the literature since it could not solve the relevant mathematical difficulties existing in the approach of anomalous phenomena such as Lévy superdiffusion [11]. The second choice is a version with unnormalized expectation. The authors in reference [7] applied this constraint into a grand canonical ensemble and a generalized quantum distribution function was derived with the help of dilute gas assumption (i.e. "factorization approximation"). The deviation of this approximation can be neglected except the temperature interval called "forbidden zone of the dilute approximation (FZDA)" [15]. As a matter of fact, most of the practical systems are far away from the FZDA, so the factorization approximation can be used to study the thermostatistic properties of nonextensive systems. However, it has been found that the values of q in nonextensive systems may not be arbitrary [16]. This point will be further proved in this paper. The third choice is called the normalized expectation and an approximate quantum distribution function was derived when q is close to one [17]. In addition, a new quantum distribution function was derived on the basis of the framework called "Incomplete Statistics (IS)" [18–20]. It is formally slightly different from that derived in reference [7]. However, in the IS framework, a complex system has a fractal phase space, the ergodic mixing hypothesis was broken down and the integral in Eucild space inevitably failed, so that the questions in the study become very complex and sometimes can not be solved

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even. Thus, at present it is still an open question for the choice of quantum distribution functions in the investigation on the thermostatistic properties of nonextensive systems.

In the present paper, the generalized Fermi-Dirac (FD) distribution function [7] derived from the Tsallis' entropy will be used to study the thermostatistic properties of a q-fermion system. Some important thermodynamic quantities such as the total number of particle, total energy and heat capacity at constant volume are derived and used to discuss the relevant characteristics of the system. The rest of this paper is organized as follows. In Section 2, expressions of several important parameters are presented. In Section 3, the thermostatistic properties of the system for two different cases of $q \leq 1$ and $q \geq 1$ are analyzed in detail, respectively. In Section 4, some interesting cases are discussed and it is pointed out that the thermostatistic properties of an original Fermi system may be directly derived as long as one chooses $q = 1$. Finally, some important conclusions are summarized.

2 The total particle number and energy of the system

It is well known that the q-generalized statistics characterized by a parameter q relies on the so-called Tsallis' entropy [3]:

$$S_q = k\frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}, \quad (1)$$

where $q \in R$, k is the Boltzmann constant, $\{p_i\}$ is a set of normalized probabilities, and W is the total number of states of the system. Within the approximation method called factorization approach, the generalized FD distribution can be written in an important formalism as [7]

$$n_q = \frac{1}{[1 + (q-1)\beta(\varepsilon - \mu)]^{1/(q-1)} + 1}, \quad (2)$$

where n_q is the average occupation number at a state with energy, ε, $\beta = 1/kT$, T is the absolute temperature, and μ is the chemical potential of the system. When $q = 1$, equation (2) becomes the well-known FD distribution. According to the Pauli principle and equation (2), one has

$$1 + (q-1)\beta(\varepsilon - \mu) \geq 0 \quad (3)$$

for an arbitrary q. From equation (3), we obtain

$$\varepsilon \begin{cases} \leq \mu + \frac{1}{(1-q)\beta} & (q \leq 1) \\ \geq \mu - \frac{1}{(q-1)\beta} & (q \geq 1). \end{cases} \quad (4)$$

On the other hand, it is easily seen from the expression $[1 + (q-1)\beta(\varepsilon - \mu)]^{1/(q-1)}$ that for an arbitrary q, the expression is smaller than 1 when $\varepsilon - \mu < 0$ and is larger than 1 when $(\varepsilon - \mu) > 0$. Using the relation $[1 + (q-1)\beta(\varepsilon - \mu)]^{1/(q-1)} = Z_q^{-1}[1 + (q-1)\beta Z_q^{q-1}\varepsilon]^{1/(q-1)}$ and equation (2), we obtain

$$n_q = \begin{cases} \sum_{j=0}^{\infty} (-1)^j z_q^{-j}[1 + (q-1)\beta z_q^{q-1}\varepsilon]^{\frac{j}{q-1}} & (\varepsilon < \mu) \\ 1/2 & (\varepsilon = \mu) \\ \sum_{j=1}^{\infty} (-1)^{j-1} z_q^{j}[1 + (q-1)\beta z_q^{q-1}\varepsilon]^{\frac{j}{1-q}} & (\varepsilon > \mu) \end{cases} \quad (5)$$

where

$$z_q = [1 + (1-q)\beta\mu]^{1/(1-q)} \quad (6)$$

is called the q-generalized fugacity [8].

Now let's consider an n-dimensional q-generalized free Fermi system. The relation between the energy and the momentum of a single particle in the system is given by

$$\varepsilon = ap^s, \quad (7)$$

where a and s are two positive constants to describe the kinetic energy of the system. For example, $a = 1/(2m)$, $s = 2$ and $a = c$, $s = 1$ correspond to the non-relativistic and ultra-relativistic cases, respectively, where m is the particle mass and c is the speed of light.

For such a system, the total particle number and total energy are, respectively, given by

$$\begin{aligned} N &= \frac{g}{h^n}\int n_q d^n r d^n p \\ &= \frac{gV}{h^n}\int \frac{d^n p}{[1 + (q-1)\beta(\varepsilon - \mu)]^{1/(q-1)} + 1} \\ &= \frac{gV}{h^n}\frac{\pi^{n/2} n/s}{\Gamma(n/2+1)} a^{-n/s} I_{n/s-1} \end{aligned} \quad (8)$$

and

$$E = \frac{g}{h^n}\int n_q \varepsilon d^n r d^n p = \frac{gV}{h^n}\frac{\pi^{n/2} n/s}{\Gamma(n/2+1)} a^{-n/s} I_{n/s}, \quad (9)$$

where g is the degenerate number, h is the Planck constant, V is the volume of the container, the parameter I_λ is given by

$$I_\lambda = \int \frac{\varepsilon^\lambda d\varepsilon}{[1 + (q-1)\beta(\varepsilon - \mu)]^{1/(q-1)} + 1}$$

$$= \begin{cases} \sum_{j=0}^{\infty} (-1)^j z_q^{-j} \int [1 + (q-1)\beta z_q^{q-1}\varepsilon]^{\frac{j}{q-1}} \varepsilon^\lambda d\varepsilon & (\varepsilon < \mu) \\ \int_\mu^\mu \frac{\varepsilon^\lambda d\varepsilon}{2} = 0 & (\varepsilon = \mu) \\ \sum_{j=1}^{\infty} (-1)^{j-1} z_q^{j} \int [1 + (q-1)\beta z_q^{q-1}\varepsilon]^{\frac{j}{1-q}} \varepsilon^\lambda d\varepsilon & (\varepsilon > \mu) \end{cases}, \quad (10)$$

and $\lambda = n/s - 1$ and $\lambda = n/s$ correspond to equations (8) and (9), respectively. It should be pointed out that equation (10) is directly derived from equations (5), (8) and (9) and the lower and upper bounds of the integrals in equation (10) depend not only on the values of μ but also on the relation between ε and μ and the different cases of $q \leq 1$ and $q \geq 1$. For example, when $\mu < 0$, it is only necessary to calculate the third integral in equation (10) because of the requirement of $\varepsilon \geq 0$. When $\mu \geq 0$, we have to calculate the three integrals in equation (10) at the same time. When $q \leq 1$ or $q \geq 1$, we must use, respectively, the first or second relation in equation (4) to determine the lower and upper bounds of the integrals in equation (10). The detail discussion will be given in Section 3

Using equation (9) and the definition of the heat capacity at constant volume

$$C_{V,q} = \left(\frac{\partial E}{\partial T}\right)_V, \tag{11}$$

one can calculate the heat capacity at constant volume of the system.

3 General properties of the system

It is clear that the energyε of a single particle in the system is one nonnegative real number, i.e.,

$$\varepsilon \geq 0, \tag{12}$$

and is constrained by the different conditions when q is different. Thus, it is necessary to discuss the thermostatistic properties of the system for two different cases of $q \leq 1$ and $q \geq 1$, respectively.

(1) The case of $q \leq 1$

From equations (4) and (12), we obtain the following relation

$$0 \leq \varepsilon \leq \mu + 1/[(1-q)\beta]. \tag{13}$$

When $\mu < 0$, $1 + 1/(\mu\beta) \leq q \leq 1$ and equation (10) may be written as

$$\begin{aligned} I_\lambda =& \sum_{j=1}^{\infty} (-1)^{j-1} z_q^j \int_0^{\mu+\frac{1}{(1-q)\beta}} [1 + (q-1)\beta z_q^{q-1}\varepsilon]^{\frac{j}{1-q}} \varepsilon^\lambda d\varepsilon \\ =& \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z_q^j}{[(1-q)\beta z_q^{q-1}]^{\lambda+1}} \frac{\Gamma(\frac{j}{1-q}+1)\Gamma(\lambda+1)}{\Gamma(\frac{j}{1-q}+\lambda+2)}. \end{aligned} \tag{14}$$

Using equations (8), (9), (11), and (14), we obtain the total particle number, total energy and heat capacity at constant volume as

$$N = \frac{gV}{h^n} \frac{\pi^{n/2}\Gamma(\frac{n}{s}+1)}{\Gamma(n/2+1)} \left(\frac{kT}{a}\right)^{n/s} f_{q,n/s}(z_q), \tag{15}$$

$$E = \frac{n}{s} NkT \frac{f_{q,n/s+1}(z_q)}{f_{q,n/s}(z_q)}, \tag{16}$$

and

$$\begin{aligned} C_{V,q} &= \left(\frac{\partial E}{\partial T}\right)_V \\ &= \frac{n}{s} Nk \left[\left(1+\frac{n}{s}\right) \frac{f_{q,n/s+1}(z_q)}{f_{q,n/s}(z_q)} - \frac{n}{s} \frac{f_{q,n/s}(z_q)}{f_{q,n/s-1}(z_q)}\right], \end{aligned} \tag{17}$$

where the generalized fermi integral

$$f_{q,D}(z_q) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z_q^{j+(1-q)D}}{(1-q)^D} \frac{\Gamma(\frac{j}{1-q}+1)}{\Gamma(\frac{j}{1-q}+D+1)}. \tag{18}$$

When $\mu \geq 0$, equation (10) may be expressed as

$$\begin{aligned} I_\lambda =& \sum_{j=0}^{\infty} (-1)^j z_q^{-j} \int_0^\mu \\ & \times [1 + (q-1)\beta z_q^{q-1}\varepsilon]^{j/(q-1)} \varepsilon^\lambda d\varepsilon + \int_\mu^\mu \frac{\varepsilon^\lambda}{2} d\varepsilon \\ & + \sum_{j=1}^{\infty} (-1)^{j-1} z_q^j \int_u^{\mu+\frac{1}{(1-q)\beta}} \\ & \times [1 + (q-1)\beta z_q^{q-1}\varepsilon]^{\frac{j}{1-q}} \varepsilon^\lambda d\varepsilon \\ =& \sum_{j=0}^{\infty} \frac{(-1)^j z_q^{-j}}{[(1-q)\beta z_q^{q-1}]^{\lambda+1}} \frac{H[\frac{j}{1-q}, \lambda+1, \lambda+2, 1-z_q^{q-1}]}{\frac{n}{s}(\frac{1}{1-z_q^{q-1}})^{\lambda+1}} \\ & + \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z_q^j}{[(1-q)\beta z_q^{q-1}]^{\lambda+1}} \\ & \times \frac{(z_q^{q-1})^{\frac{j}{1-q}+1} H[-\lambda, \frac{j}{1-q}+1, \frac{j}{1-q}+2, z_q^{q-1}]}{\frac{j}{1-q}+1}, \end{aligned} \tag{19}$$

where

$$H[a, b, b+1, C] = b \int_0^1 (1 - Ct)^{-a} t^{b-1} dt \tag{20}$$

is called the hyper-geometric function, and a, b and C are some parameters which are independent of t.

Substituting equation (19) into equations (8), (9) and (11), one can obtain

$$N = \frac{gV}{h^n} \frac{\pi^{n/2} n/s}{\Gamma(n/2+1)} \left(\frac{kT}{a}\right)^{n/s} f_{q,n/s}(z_q), \tag{21}$$

$$E = NkT \frac{f_{q,n/s+1}(z_q)}{f_{q,n/s}(z_q)}, \tag{22}$$

and

$$\frac{C_{V,q}}{Nk} = (1 + n/s) \frac{f_{q,n/s+1}(z_q)}{f_{q,n/s}(z_q)} - n/s \frac{\partial f_{q,n/s+1}(z_q)/\partial z_q}{\partial f_{q,n/s}(z_q)/\partial z_q}, \tag{23}$$

where the q-generalized Fermi integral

$$\begin{aligned} f_{q,D}(z_q) =& \sum_{j=0}^{\infty} \frac{(-1)^j z_q^{-j+(1-q)D}}{(1-q)^D} \\ & \times \frac{(1-z_q^{q-1})^D H[\frac{j}{1-q}, D, 1+D, 1-z_q^{q-1}]}{D} \\ & + \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z_q^{(1-q)(D-1)}}{(1-q)^D} \\ & \times \frac{H[1-D, \frac{j}{1-q}+1, \frac{j}{1-q}+2, z_q^{q-1}]}{\frac{j}{1-q}+1}. \end{aligned} \tag{24}$$

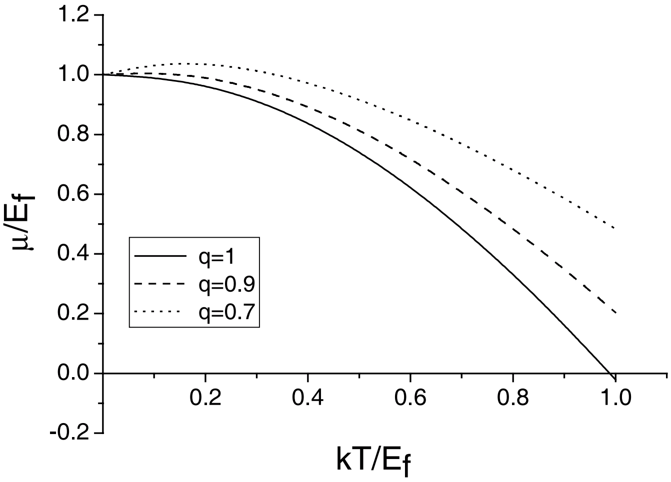

Fig. 1. The chemical potential versus temperature curves of a nonextensive ideal Fermi gas for $q \leq 1$.

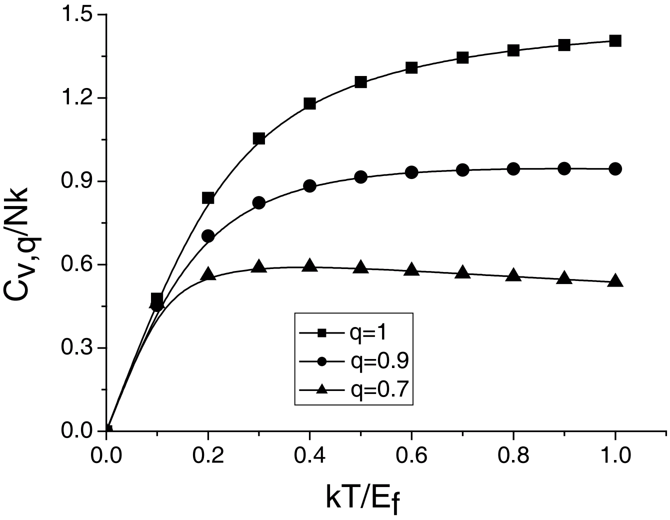

Fig. 2. The heat capacity versus temperature curves of a nonextensive ideal Fermi gas for $q \leq 1$.

It is seen from equation (13) that in the limit of $T \to 0$ K, the energy of particles larger than chemical potential is not allowed, that is to say, all the energy levels below the Fermi energy level E_F (i.e. chemical potential μ at zero temperature) are filled by particles while all the energy levels above the Fermi energy level E_F are blank. From equations (2) and (13), one can get

$$n_q = \begin{cases} 1 & (\varepsilon < E_F) \\ 1/2 & (\varepsilon = E_F). \end{cases} \tag{25}$$

By using equation (25), equation (8) may be expressed as

$$\begin{aligned} N &= \frac{gV}{h^n} \frac{\pi^{n/2} n/s}{\Gamma(n/2+1)} a^{-n/s} \\ &\quad \times \left\{ \int_0^{E_F} \varepsilon^{n/s-1} d\varepsilon + \int_{E_F}^{E_F} \frac{\varepsilon^{n/s-1}}{2} d\varepsilon \right\} \\ &= \frac{gV}{h^n} \frac{\pi^{n/2}}{\Gamma(n/2+1)} a^{-n/s} E_F^{n/s}. \end{aligned} \tag{26}$$

From equation (26), one can determine the Fermi energy of the system as

$$E_F = \left[\frac{N h^n a^{n/s} \Gamma(n/2+1)}{gV \pi^{n/2}} \right]^{s/n}, \tag{27}$$

which is independent of q and is the same as that of an original ideal Fermi gas.

From equations (15), (21) and (27) and the definition of z_q, one can numerically calculate the generalized fugacity z_q when the dimensionless temperature kT/E_F is given and generate the curves of the chemical potential varying with the dimensionless temperature kT/E_F for different q values, as shown in Figure 1, where the parameters $n = 3$ and $s = 2$ are chosen, which correspond to the case of a nonrelativistic ideal nonextensive Fermi system in three-dimensional space. It is seen from Figure 1 that the chemical potentials of $q < 1$ are different from those of $q = 1$. The more the q value is away from 1, the larger the difference between them. The difference increases with temperature. When $T \to 0$ K, the difference will disappear and all the chemical potentials for the various systems with different q values are equal to the Fermi energy. Figure 1 also shows that the chemical potential μ of $q = 1$ is equal to the Fermi energy at $T \to 0$ K and monotonically decreases with the increase of temperature, while the chemical potential μ of $q < 1$ is not a monotonic function of temperature and may be larger than the Fermi energy in certain region of temperature.

Similarly, equations (17) and (23) can be used to plot the curves of the heat capacity at constant volume varying with the dimensionless temperature kT/E_F for different q values, as shown in Figure 2. It is seen from the curves in Figure 2 that the heat capacity will tend to zero when temperature approaches to absolute zero no matter what the q value is, and the third law of thermodynamics still holds. At very low temperatures, the difference between the heat capacities for the systems of $q < 1$ and $q = 1$ is not obvious. The difference increases with temperature. The heat capacity for the system of $q = 1$ is a monotonically increasing function of temperature, while the heat capacity for the system of $q < 1$ is not a monotonic function of temperature. However, unlike the heat capacity of the nonextensive Bose system [8,10] which has a phase transition point and the heat capacity at the critical temperature of BEC may be discontinuous, the heat capacity of the q-generalized Fermi system continuously varies with temperature. It first increases and then decreases as temperature is increased so that there is a maximum heat capacity at some value of temperature. The main cause is that the fermions must be constrained by the Pauli principle. Figure 2 also shows that for the same temperature, the heat capacity monotonically decreases with the decrease of the q value. It means that the interaction of particles in the nonextensive fermion system is attractive when q is less than 1 and that the less energy is needed to excite the particles in the system of $q < 1$ than in the system of $q = 1$. This coincides with the case of the nonextensive Bose system [8].

(2) The case of $q \geq 1$

When $\mu \leq 0$, $0 \leq \varepsilon < \infty$ and equation (10) may be expressed as

$$I_\lambda = \sum_{j=1}^{\infty} (-1)^{j-1} z_q^j \int_0^\infty [1 + (q-1)\beta z_q^{q-1}\varepsilon]^{\frac{j}{1-q}} \varepsilon^\lambda d\varepsilon$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z_q^j}{[(q-1)\beta z_q^{q-1}]^{\lambda+1}} \frac{\Gamma(\frac{j}{q-1} - \lambda - 1)\Gamma(\lambda+1)}{\Gamma(\frac{j}{q-1})}. \quad (28)$$

In order to guarantee the integral to be larger than zero, the condition $\frac{j}{q-1} > \lambda + 1$, i.e., $q < \frac{\lambda+2}{\lambda+1}$ must be satisfied. Substituting equation (28) into equations (8), (9), and (11), we find that the total particle number, total energy and heat capacity at constant volume are the same as equations (15–17), while in this case the q-generalized Fermi integral may be expressed as follows

$$f_{q,D}(z_q) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z_q^{j-(q-1)D}}{(q-1)^D} \frac{\Gamma(\frac{j}{q-1} - D)}{\Gamma(\frac{j}{q-1})}. \quad (29)$$

When $0 < \mu \leq 1/[\beta(q-1)]$, $1 < q \leq 1 + 1/(\mu\beta)$ and $0 \leq \varepsilon < \infty$. Equation (10) may be expressed as

$$I_\lambda = \sum_{j=0}^{\infty} (-1)^j z_q^{-j} \times \int_0^\mu [1 + (q-1)\beta z_q^{q-1}\varepsilon]^{j/(q-1)} \varepsilon^\lambda d\varepsilon + \int_\mu^\mu \frac{\varepsilon^\lambda}{2} d\varepsilon$$
$$+ \sum_{j=1}^{\infty} (-1)^{j-1} z_q^j \int_u^\infty [1 + (q-1)\beta z_q^{q-1}\varepsilon]^{\frac{j}{1-q}} \varepsilon^\lambda d\varepsilon$$
$$= \sum_{j=0}^{\infty} \frac{(-1)^j z_q^{-j}}{[(q-1)\beta z_q^{q-1}]^{\lambda+1}} \times \frac{(z_q^{q-1} - 1)^{\lambda+1} H[\frac{j}{1-q}, \lambda+1, \lambda+2, 1 - z_q^{q-1}]}{\lambda+1}$$
$$+ \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z_q^j}{[(q-1)\beta z_q^{q-1}]^{\lambda+1}} \times \frac{H\left[\frac{j}{q-1}, \frac{j}{q-1} - \lambda - 1, \frac{j}{q-1} - \lambda, \frac{1}{1-z_q^{q-1}}\right]}{(\frac{j}{q-1} - \lambda - 1)(z_q^{q-1} - 1)^{\frac{j}{q-1} - \lambda - 1}}, \quad (30)$$

where $q < \frac{\lambda+2}{\lambda+1}$ must be also satisfied. The expressions of N, E, and $C_{V,q}$ are still given by the same forms as equations (21–23), while the q-fermi integral is given by

$$f_{q,D}(z_q) = \sum_{j=0}^{\infty} \frac{(-1)^j z_q^{-j-(q-1)D}}{(q-1)^D} \times \frac{(z_q^{q-1} - 1)^D H[\frac{j}{1-q}, D, D+1, 1 - z_q^{q-1}]}{D}$$
$$+ \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z_q^{j-(q-1)D}}{(q-1)^D} \times \frac{(z_q^{q-1} - 1)^{D - \frac{j}{q-1}} H\Big[\frac{j}{q-1}, \frac{j}{q-1} - D, \frac{j}{q-1} - D + 1, \frac{1}{1-z_q^{q-1}}\Big]}{(\frac{j}{q-1} - D)}. \quad (31)$$

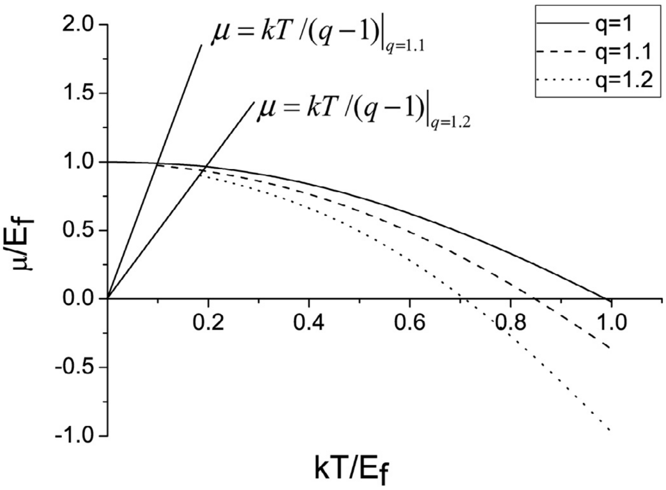

Fig. 3. The chemical potential versus temperature curves of a nonextensive ideal Fermi gas for $q \geq 1$.

When $1/[\beta(q-1)] < \mu$, it is seen from equation (4) that there is a constrained condition $\varepsilon > 0$, while $\varepsilon = 0$ is unallowable. It is in contradiction with equation (12). On the other hand, it is seen from equation (6) that when $1/[\beta(q-1)] < \mu$, z_q may become an imaginary number. Just like the cut-off in the probability distribution function derived from the Tsallis entropy [11], there are a cut-off in the generalized fugacity and a corresponding cut-off in the chemical potential for a nonextensive free ideal fermi gas, as shown in Figure 3, where the straight line stands for the critical condition of the cut-off. Figure 3 shows that the curves of the chemical potential varying with temperature are only allowed to be situated in the right side of the straight line which has a slope of $kT/(q-1)$. Obviously, the slope of the straight line increases with the decrease of q and the cut-off temperature decreases with the increase of the slope of the straight line. When q tends to 1, the slope of the straight line becomes infinite, the cut-off temperature tends to zero, and the chemical potential is equal to the Fermi energy. It implies the fact that the case of $1/[\beta(q-1)] < \mu$ is not allowed for the systems of $q > 1$.

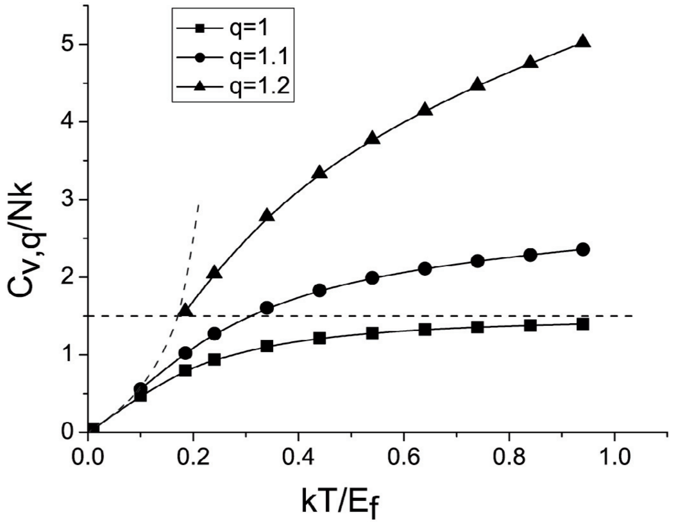

Fig. 4. The heat capacity versus temperature curves of a nonextensive ideal Fermi gas for $q \geq 1$.

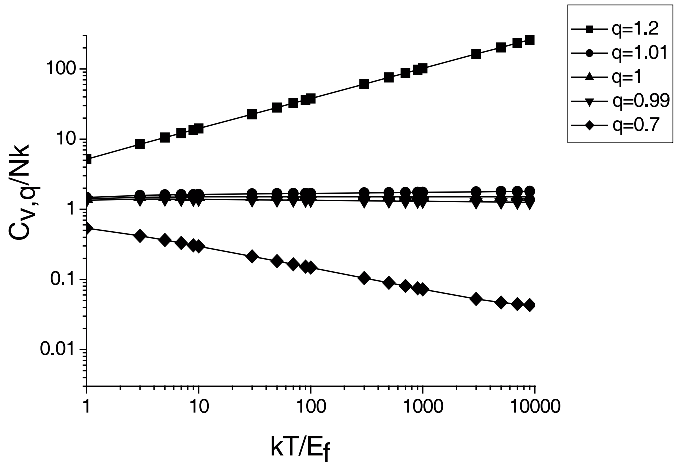

Fig. 5. The heat capacity of a nonextensive ideal Fermi gas at high temperatures.

It is seen from the curves in Figure 3 that the cut-off of the chemical potential of a nonextensive free ideal fermi gas is directly dependent on the parameter q and the chemical potentials of $q > 1$ are always less than the Fermi energy. At the same temperature, the chemical potential of $q > 1$ are smaller than that of $q = 1$. The more the q value is away from 1, the larger the difference between them. The difference increases with temperature.

Corresponding to the cut-off temperature, there is a cut-off of the heat capacity for a free q-generalized Fermi system, as shown in Figure 4, where the dash curve represents the different cut-off points for different q. At the same temperature, the heat capacity in the system of $q > 1$ is larger than that of $q = 1$. The difference between them increases with temperature. The heat capacity in the system of $q > 1$ is divergent as temperature tends to infinite. This is similar to the case of the generalized Bose system with $q > 1$ [8,10].

4 Discussion

At high temperatures, the heat capacity of the generalized ideal Fermi gas does not tend to a constant. When $q > 1$, the heat capacity increases with temperature; when $q < 1$, the heat capacity decreases with temperature; as shown in Figure 5. This is very abnormal but coincides with the case of the generalized Bose system [8,10]. It implies the fact that at high temperatures, the quantum effects of the generalized Bose and Fermi systems are negligible and consequently their thermostatistic properties tend to unanimity.

When $q \to 1$, equation (2) becomes the well-known FD distribution function. The q-generalized Fermi integral is simplified as the original Fermi integral, so that the familiar results in textbooks [21] may be directly derived from the above equations. For example, we can easily obtain some thermodynamic parameters of an original Fermi system as

$$N = \frac{gV}{h^n} \frac{\pi^{n/2} \Gamma(\frac{n}{s}+1)}{\Gamma(n/2+1)} \left(\frac{kT}{a}\right)^{n/s} f_{n/s}(z), \tag{32}$$

$$E = \frac{n}{s} NkT \frac{f_{n/s+1}(z)}{f_{n/s}(z)}, \tag{33}$$

and

$$\begin{aligned} C_V &= \left(\frac{\partial E}{\partial T}\right)_V \\ &= \frac{n}{s} Nk \left[\left(1+\frac{n}{s}\right) \frac{f_{n/s+1}(z)}{f_{n/s}(z)} - \frac{n}{s} \frac{f_{n/s}(z)}{f_{n/s-1}(z)} \right], \end{aligned} \tag{34}$$

where the fugacity $z = e^{\beta\mu}$ and the original Fermi integral

$$f_D(z) = \frac{1}{\Gamma(D)} \int_0^\infty \frac{x^{D-1} dx}{z^{-1} e^x + 1}. \tag{35}$$

When the temperature is very high, $z \ll 1$ and the Fermi integral can be expanded as [22]

$$f_D(z) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{j^D}, \tag{36}$$

while at low temperatures, equation (35) may be written as

$$\begin{aligned} f_D(z) = \frac{(\ln z)^D}{\Gamma(D+1)} &\times \left[1 + \sum_{j=2,4,6...} \left\{ 2D(D-1)\cdots(D+1-j) \right.\right. \\ &\left.\left. \times \left(1 - \frac{1}{2^{j-1}}\right) \frac{\varsigma(j)}{(\ln z)^j} \right\} \right], \end{aligned} \tag{37}$$

which is just the desired asymptotic formula–commonly known as Sommerfeld's lemma [21], where $\varsigma(j)$ is the Rimman-Zita function.

5 Conclusions

By using the nonextensive FD distribution function derived from the Tsallis' entropy and introducing some interesting quantities such as the generalized Fermi integral and the hyper-geometric function, the expressions of some important thermodynamic parameters for a q-generalized Fermi system with the general energy spectrum $\varepsilon = ap^s$ are derived analytically. The parameters include the total particle number, total energy, specific heat at constant volume, Fermi energy, and so on. It is found that most of the thermodynamic properties of a q-generalized Fermi system are closely dependent on the value of q. When $q < 1$, the chemical potential of a q-generalized Fermi system in certain region of temperature may be larger than the Fermi energy. When $q > 1$, the chemical potential of a q-generalized Fermi system is always smaller than the Fermi energy and there exist some natural constraints (cut-off) for some thermodynamic parameters. At low temperatures, the thermostatistic properties of the system are very different from those of the generalized Bose system. However, the abnormal behavior of the heat capacity of the system at high temperatures coincides with that of the q-generalized Bose system. The general properties of the system are discussed in detail. The results obtained here are general, from which the thermostatistic properties of an original Fermi system may be directly derived.

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